A shortened recurrence relation for the Bernoulli numbers

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Abstract

In this note, starting with a little-known result of Kuo, I derive a recurrence relation for the Bernoulli numbers B_{2n}, n being any positive integer. This new recurrence seems advantageous in comparison to other known formulae since it allows the computation of both B_{4n} and B_{4n+2} from only $B_0, B_2, \ldots, B_{2n}$.

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The Bernoulli numbers B_n, n being a nonnegative integer, can be defined by the generating function

$$\frac{x}{\mathrm{e}^x - 1} = \sum_{n=0}^{\infty} B_n \, \frac{x^n}{n!} \, , \quad |x| < 2\pi. \tag{1}$$

The first few values are well-known: $B_0 = 1$, $B_1 = -\frac{1}{2}$, and $B_2 = \frac{1}{6}$. It is also well-known that $B_n = 0$ for odd values of n, $n > 1$. For even values

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of n $(= 2m)$, $n > 1$, the numbers B_{2m} form a subsequence of non-null real numbers such that $(-1)^{m+1} B_{2m} > 0$. In other words, the entire sequence of Bernoulli numbers up to any B_n, $n > 1$, consists of B_0 and B_1, given above, and the preceding numbers B_{2m}, $m = 1, \ldots, \lfloor n/2 \rfloor$. The basic properties of these numbers can be found in Sec. 9.61 of [1].

The numbers B_n appear in many instances in pure and applied mathematics, most notably in number theory, finite differences calculations, and asymptotic analysis. Therefore, the efficient computation of the numbers B_n is of great interest. To this end, recurrence formulae were soon recognized as the most efficient tool [2]. One of the simplest such relations is found by multiplying both sides of Eq. (1) by $\mathrm{e}^x - 1$, using the Cauchy product with the Maclaurin series for $\mathrm{e}^x - 1$, and equating the coefficients of the powers of x, which results in

$$\sum_{j=0}^{n} \binom{n+1}{j} B_j = 0\,, \quad n \geq 1\,. \tag{2}$$

This kind of recurrence formula has the disadvantage of demanding the previous knowledge of all $B_0, B_1, \ldots, B_{n-1}$ for the computation of B_n. In searching for more efficient formulae, shortened recurrence relations of two different types have been discovered. The first type consists of the so-called lacunary recurrence relations, in which B_n is determined only from every second, or every third, etc., preceding Bernoulli numbers (see, e.g., the lacunary formula by Ramanujan [3]). The second type demands the knowledge of only the second-half of the Bernoulli numbers up to B_{n-1} in order to compute B_n [4]. For an extensive study of these and other recurrence relations involving the Bernoulli numbers, see [5].

Here in this note, I apply the Euler's formula relating the even zeta value $\zeta(2n)$ to B_{2n} to a little-known recurrence formula for $\zeta(2n)$ obtained by Kuo [6], in order to convert it into a recurrence formula for B_{2n}. By doing this, in fact I introduce a third type of recurrence relation for the Bernoulli numbers, in the sense that it allows us to compute both B_{4n} and B_{4n+2} from only the first-half of the preceding numbers, i.e. $B_0, B_2, \ldots, B_{2n}$. Then, the efficiency of this new recurrence formula certainly surpasses that of most known formulae, mainly for large values of n.

For real values of s, $s > 1$, the Riemann zeta function is defined as $\zeta(s) := \sum_{n=1}^{\infty} 1/n^s$. In this domain, the convergence of this series is guaranteed by the integral test.[1] For positive even values of s, one has the well-known Euler's formula (1740) [7]:

$$\zeta(2n) = (-1)^{n-1} \, \frac{2^{2n-1} \, B_{2n}}{(2n)!} \, \pi^{2n} \, , \tag{3}$$

which, in face of the rationality of every Bernoulli number B_n, yields $\zeta(2n)$ as a rational multiple of π^{2n}. The function $\zeta(s)$, as defined above, can be extended to the entire complex plane (except the only simple pole at $s = 1$) by analytic continuation, which yields $\zeta(0) = -\frac{1}{2}$.[2] The reader should note that Eq. (3) remains valid for $n = 0$.

These are the necessary ingredients to state our first lemma, which comes

[1]For $s = 1$, one has the harmonic series $\sum_{n=1}^{\infty} 1/n$, which diverges to infinity.

[2]Alternatively, we can use the globally convergent series $\frac{1}{1-2^{1-s}} \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{(k+1)^s}$, due to Hasse (1930) [8], which is valid for all complex numbers $s \neq 1 + \frac{2\pi m}{\ln 2}\,\mathrm{i}$, m being any integer. At $s = 0$, it reduces to $\zeta(0) = -\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \sum_{k=0}^{n} (-1)^k \binom{n}{k} = -\frac{1}{2} - \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{2^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k}$. Of course, $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$ for all $n > 0$, so $\zeta(0) = -\frac{1}{2}$.

from a little-known recurrence formula by Kuo (1949) [6], just written directly in terms of the Riemann zeta function.

Lemma 1 (Kuo's recurrence formula for $\zeta(2n)$). *For any positive integer n, one has*

$$\zeta(2n) = \frac{2^{2n-1}\,\pi^{2n}}{4\,(n-1)!^2\,(2n-1)} + \frac{1}{(n-1)!}\,\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k\,\frac{\zeta(2k)\,(2\pi)^{2n-2k}}{(n-2k)!\,(2n-2k)}$$
$$+\frac{1}{\pi}\,\sum_{k=0}^{\lfloor n/2 \rfloor}\,\sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^{k+j}\,\zeta(2k)\,\zeta(2j)\,\frac{(2\pi)^{2n-2k-2j+1}}{(n-2k)!\,(n-2j)!\,(2n-2k-2j+1)}\,.$$

This recurrence formula is proved in [6] by developing successive integrations (from 0 to x) of the Fourier series $\sum_{n=1}^{\infty} \sin{(n\,x)}/n = (\pi - x)/2$, which converges for all positive real $x < 2\,\pi$, and then applying the Parseval's theorem.

We are now in a position to prove the following theorem.

Theorem 1 (Recurrence relation for B_{2n}). *For any positive integer n, one has*

$$B_{2n} = (-1)^{n-1}\left[a_n - b_n\,\sum_{k=0}^{\lfloor n/2 \rfloor} \frac{B_{2k}}{(2k)!\,(n-2k)!\,(n-k)} \right.$$
$$\left. +(2n)!\,\sum_{k=0}^{\lfloor n/2 \rfloor} \frac{B_{2k}}{(2k)!\,(n-2k)!}\,\sum_{j=0}^{\lfloor n/2 \rfloor} \frac{B_{2j}}{(2j)!\,(n-2j)!}\,\frac{1}{2n-2k-2j+1}\right], \qquad (4)$$

where $a_n = \frac{n}{2}\,\frac{(2n-2)!}{(n-1)!^2}$ *and* $b_n = \frac{(2n)!}{2\,(n-1)!}$.

PROOF. By dividing both sides of the Kuo's recurrence formula, as given in Lemma 1, by π^{2n}, one has

$$\frac{\zeta(2n)}{\pi^{2n}} = \frac{2^{2n-1}\,\pi^{2n}}{4\,(n-1)!^2\,(2n-1)} + \frac{1}{(n-1)!}\,\sum_{k=0}^{\lfloor n/2\rfloor}(-1)^k\,\frac{\zeta(2k)}{\pi^{2k}}\,\frac{2^{2n-2k}}{(n-2k)!\,(2n-2k)}$$
$$+\sum_{k=0}^{\lfloor n/2\rfloor}\sum_{j=0}^{\lfloor n/2\rfloor}(-1)^{k+j}\,\frac{\zeta(2k)}{\pi^{2k}}\,\frac{\zeta(2j)}{\pi^{2j}}\,\frac{2^{2n-2k-2j+1}}{(n-2k)!\,(n-2j)!\,(2n-2k-2j+1)}\,. \quad (5)$$

From Euler's equation, Eq. (3), one knows that $\frac{\zeta(2m)}{\pi^{2m}} = (-1)^{m-1}\,\frac{2^{2m-1}\,B_{2m}}{(2m)!}$, which is valid for all integer m, $m \ge 0$. By substituting this in Eq. (5), one finds, after some algebra,

$$\frac{B_{2n}}{(2n)!} = \frac{(-1)^{n-1}}{4\,(n-1)!^2\,(2n-1)} - \frac{(-1)^{n-1}}{(n-1)!}\,\sum_{k=0}^{\lfloor n/2\rfloor}\frac{B_{2k}}{(2k)!}\,\frac{1}{(n-2k)!\,(2n-2k)}$$
$$+(-1)^{n-1}\sum_{k=0}^{\lfloor n/2\rfloor}\sum_{j=0}^{\lfloor n/2\rfloor}\frac{B_{2k}}{(2k)!}\,\frac{B_{2j}}{(2j)!}\,\frac{1}{(n-2k)!\,(n-2j)!\,(2n-2k-2j+1)}\,. \quad (6)$$

Now, put $(-1)^{n-1}$ in evidence and multiply both sides by $(2n)!$. This yields

$$B_{2n} = (-1)^{n-1}\left[\frac{(2n)!}{4\,(n-1)!^2\,(2n-1)} - \frac{(2n)!}{(n-1)!}\,\sum_{k=0}^{\lfloor n/2\rfloor}\frac{B_{2k}}{(2k)!}\,\frac{1}{(n-2k)!\,(2n-2k)}\right.$$
$$\left.+(2n)!\,\sum_{k=0}^{\lfloor n/2\rfloor}\sum_{j=0}^{\lfloor n/2\rfloor}\frac{B_{2k}}{(2k)!}\,\frac{B_{2j}}{(2j)!}\,\frac{1}{(n-2k)!\,(n-2j)!\,(2n-2k-2j+1)}\right., \quad (7)$$

which readily simplifies to

$$B_{2n} = (-1)^{n-1}\left[\frac{n\,(2n-2)!}{2\,(n-1)!^2} - \frac{(2n)!}{2\,(n-1)!}\,\sum_{k=0}^{\lfloor n/2\rfloor}\frac{B_{2k}}{(2k)!\,(n-2k)!\,(n-k)}\right.$$
$$\left.+(2n)!\,\sum_{k=0}^{\lfloor n/2\rfloor}\frac{B_{2k}}{(2k)!\,(n-2k)!}\,\sum_{j=0}^{\lfloor n/2\rfloor}\frac{B_{2j}}{(2j)!\,(n-2j)!}\,\frac{1}{2n-2k-2j+1}\right]. \quad (8)$$

$\square$

In fact, the auxiliary terms a_n and b_n in our Theorem 1 can be cast in a more suitable form for computational purposes. For a_n, one has $a_1 = \frac{1}{2}$ and

$$a_n = \frac{n}{2\,(n-1)!}\,\frac{(2n-2)!}{(n-1)!} = \frac{n}{2\,(n-1)!}\prod_{m=2}^{n}(2n-m)\,, \tag{9}$$

for all $n > 1$. For b_n, one has $b_1 = 1$ and

$$\begin{aligned} b_n &= \frac{(2n)\,(2n-1)\cdot\,\ldots\,\cdot n\,(n-1)!}{2\,(n-1)!} = n^2\,(2n-1)\cdot\,\ldots\,\cdot(n+1) \\ &= n^2\prod_{m=1}^{n-1}(2n-m)\,, \end{aligned} \tag{10}$$

for all $n > 1$.

Since $B_0 = 1$ and the recurrence relation in Theorem 1 has only the four basic numeric operations ($+$, $-$, $\times$, and $\div$), it is straightforward to show, by induction on n, that every B_{2n}, $n \geq 1$, is a rational number, though this is a well-known characteristic of these numbers (see, e.g., Ref. [5]).

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